International Journal of Mathematical Education in Science and Technology
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713736815

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To cite this Article: Sirotic, N. and Zazkis, R. , 'Irrational numbers on the number line - where are they?', International Journal of Mathematical Education in Science and Technology, 38:4, 477 - 488
To link to this article: DOI: 10.1080/00207390601151828
URL: http://dx.doi.org/10.1080/00207390601151828

International Journal of Mathematical Education in Science and Technology, Vol. 38, No. 4, 15 June 2007, 477–488

Irrational numbers on the number line – where are they?

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(*Received 14 July 2005*)

This paper reports part of an ongoing investigation into the understanding of irrational numbers by prospective secondary school teachers. It focuses on the representation of irrational numbers as points on a number line. In a written questionnaire, followed by a clinical interview, participants were asked to indicate the exact location of the square root of 5 on a number line. The results suggest confusion between irrational numbers and their decimal approximation and overwhelming reliance on the latter. Pedagogical suggestions are discussed.

1. Introduction

Can the *exact* location of $\sqrt{5}$ be found on the number line? In this article we consider the answers of a group of preservice secondary school teachers to this question, in light of their general conceptions of irrational numbers and their representations.

This report is part of an ongoing investigation into the understanding of irrational numbers. Previously we focused on formal and intuitive knowledge of irrationality as well as on representations of irrational numbers [1, 2]. Here we limit our focus to the geometric representation of irrational length as specified by a point on a number line.

2. Background: snapshot from research literature

Prior research on irrational numbers is rather slim. A small number of researchers who investigated students' and teachers' understanding of irrational numbers reported the difficulty that participants have in identifying the set membership, that is, recognizing numbers as either rational or irrational [3, 4], in providing appropriate definitions for rational and irrational numbers [5], and in flexible use of different representations [6].

Of particular interest here is the study of Arcavi *et al*. [7] related to their work on using the history of mathematics to design pre-service and in-service teacher courses. These researchers report several findings on teachers' knowledge, conceptions, and/or misconceptions regarding irrational numbers. One of the most striking discoveries

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International Journal of Mathematical Education in Science and Technology
ISSN 0020–739X print/ISSN 1464–5211 online © 2007 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/00207390601151828

from their study is that there is a widespread belief among teachers that irrationality relies upon decimals. This study was conducted on 84 in-service teachers who attended a summer teacher training programme related to a national mathematics curriculum for junior high schools in Israel. Arcavi *et al*. [7] report that 70% of teachers knew that the first time the concept of irrationality arose was before the Common Era (Greeks). However, although the majority knew 'when' it arose, very few also knew 'how' it arose. This became particularly apparent when they were asked to order chronologically the appearance of three concepts: negative numbers, decimal fractions, and irrationals. 55% of teachers (and an additional 10% did not answer) indicated that decimal fractions preceded irrationals in the historical development. The authors concluded that this not only indicated the lack of knowledge about the relatively recent development of decimals, but more importantly, it indicated that the origin of the concept of irrationality, although associated with the Greeks, is conceived as relying upon decimals, and not connected to geometry as occurred historically (commensurable and incommensurable lengths). Arcavi *et al*. [7] point out that 'the historical origins of irrationals in general, and the connections to geometry in particular, can provide an insightful understanding of the concept as well as teaching ideas for the introduction of the topic in the classroom' (p. 18). This particular connection to geometry is of our interest here.

As a commentary, we find it interesting to point out that the three concepts mentioned in study [7] are generally introduced to students in the reverse order from how they developed historically. The concept of irrationality received its first proper theoretical treatment by Eudoxus around 400 BC, and it appears in Euclid's *Elements* [8]. On the other hand, decimals were introduced by Simon Stevin in his *De Thiende* in 1585 [9]. Historically, the first formal introduction to negative numbers appears in *Introduction to Algebra* by Leonard Euler in 1770.

3. Research setting

3.1. *The task: show how you would find the exact location of $\sqrt{5}$ on the number line*

The task was designed in order to investigate understanding of the geometric representation of an irrational number. In particular, we were interested in what means the participants would use in order to locate $\sqrt{5}$ on the number line precisely. It is said that to every real number there corresponds exactly one point on the real number line. One may find this difficult to believe if one has never seen an irrational point located on the number line, especially considering the fact that the number line is everywhere dense with rational numbers. Further, we included $\sqrt{5}$ in this item rather than the 'generic' $\sqrt{2}$ assuming that the latter would lead some participants to automatic recall from memory, rather than construction.

The number line drawing given to students with this task was intentionally set in the Cartesian plane with a visible grid to simplify the straightedge and compass construction (i.e. there was no need to draw a perpendicular line at 2). It was intended to aid in the invoking of the Pythagorean Theorem in efforts to construct the required length. The expected response is shown in figure 1.

We were interested to see whether participants would use this conventional or similar approach or whether they would resort to thinking in terms of decimal expansions.

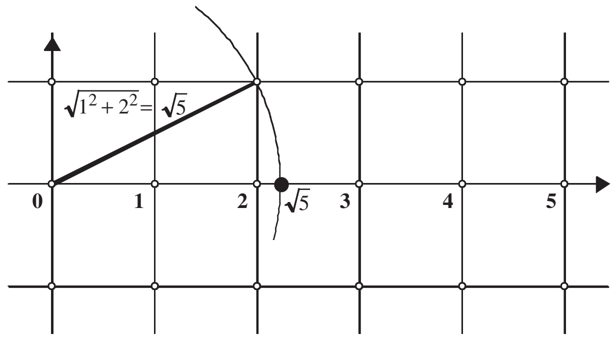

Figure 1. Geometric construction of $\sqrt{5}$.

3.2. *Participants*

Participants in this study were 46 preservice secondary school teachers enrolled in the professional development course 'Designs for Learning: Secondary Mathematics'. They responded to the item above as part of a written questionnaire that included several additional items related to irrationality. Following the completion of the questionnaire 16 volunteers from the group participated in a clinical interview in which their responses and general dispositions were probed further.

4. Analysis of responses

The geometric representation of irrational numbers was strangely absent from the concept images of many participants. The common conception of real number line appeared to be limited to rational number line, or even more strictly, to decimal rational number line where only finite decimals receive their representations as 'points on the number line'. This is in agreement with the practical experience that finite decimal approximations are both convenient and sufficient, which could be the source of these conflicts.

Table 1 summarizes the results of the written responses to the geometric construction task.

The responses fall into five distinct categories: an exact location of the point using the knowledge of Pythagorean Theorem, more or less fine decimal approximation, very rough approximation (between 2 and 3), responses related to graphing of a related function, and an outright claim that this is impossible to do. Next we exemplify and examine some representatives of each category.

4.1. *Geometric approaches*

Ten participants (out of 46) used geometric approaches, nine of which we classified as precise. We presented above what could be considered a conventional geometric approach. Indeed, it appeared in the work of four participants. This is an example of such a response:

- The length of the hypotenuse shown is $\sqrt{5}$ (see figure 2). Just rotate the segment so it falls on the number line, then move it up on the line (horizontal translation 1 unit to the left).

Table 1. Quantification of results for the geometric construction of $\sqrt{5}$ ($n = 46$).

Response category	Number of participants [%]
Exact, using Pythagorean Theorem	9 [19.6%]
Decimal approximation using one or more digits after the decimal point	18 [39.2%]
Very rough approximation, i.e. 'between 2 and 3'	6 [13%]
Other response (for example, using graphs of $f(x) = \sqrt{x}$ or $f(x) = x^2 - 5$)	6 [13%]
Responses arguing 'you can't'	4 [8.7%]
No response	3 [6.5%]

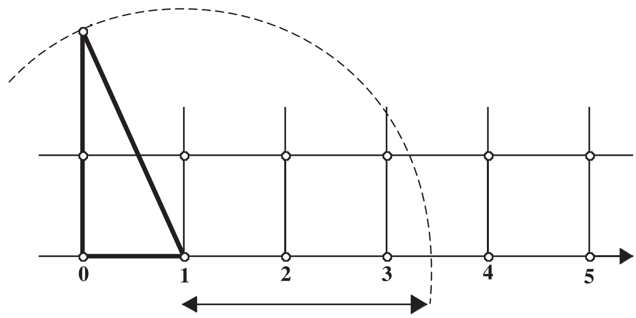

Figure 2. Alternative geometric approach to construct $\sqrt{5}$.

Two other valid geometric approaches were found. One of them is a slight variation of the previous response. Instead of determining the placement of $\sqrt{5}$ by construction it uses a 'ready made' right triangle with the side lengths of 1 and 2. Four participants gave the response such as this.

Make the hypotenuse $h = \sqrt{1^2 + 2^2} = \sqrt{5}$ lie on the number line. (See figure 3.).

The other valid geometric approach is the familiar spiral of right triangles constructed by successive applications of the Pythagorean Theorem with one of the legs always equal to 1 and the other leg equal to the hypotenuse of the previously constructed triangle. This construction, as demonstrated in figure 4, is a more generalized version of the conventional geometric approach in the sense that a square root of any whole number can be constructed in this way. It might not be the most efficient construction, but it spares one from having to think about what two perfect squares add up to the required square of the length of the hypotenuse. Only one participant used this approach.

The next response is interesting. It uses geometric approximation as shown in figure 5. The figure was accompanied by the following note: 'Area A = Area B, where A is a square. $\sqrt{5} \times \sqrt{5} = 5 \times 1'$. This solution seems to involve 'eye-balling' when the partial pieces in square A will make a whole squared unit.

4.2. *Numerical approaches*

Next we present a range of responses from the written part, arranged by the degree of accuracy. Twenty-four participants (over 52%) offered an approach based on the

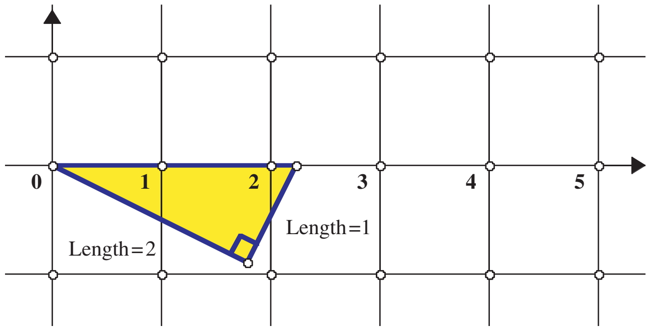

Figure 3. Locating $\sqrt{5}$ by a 'ready-made' right triangle.

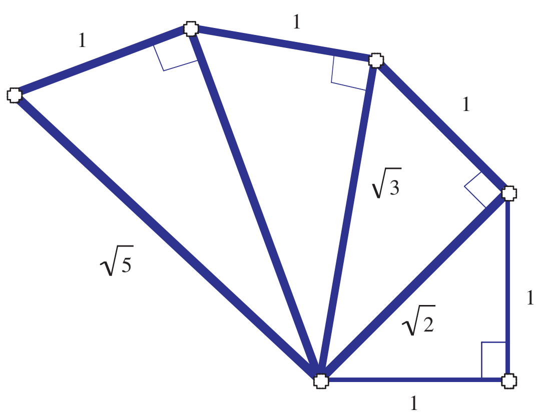

Figure 4. Construction of $\sqrt{5}$ using successive triangles.

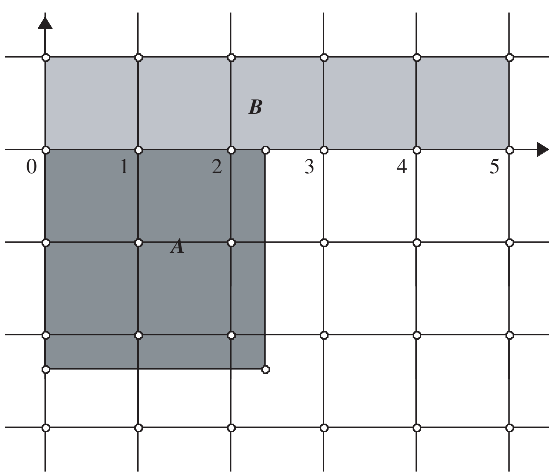

Figure 5. Locating $\sqrt{5}$ by 'eye-balling' the areas.

decimal expansion of $\sqrt{5}$. We start with those who offered a very rough approximation, and end with those who demonstrated a genuine striving for accuracy.

- Some participants circled a 'big blob' around the area of expected location and said 'somewhere around here'.
- $$\begin{array}{cccccc} \sqrt{4} & \sqrt{5} & \sqrt{6} & \sqrt{7} & \sqrt{8} & \sqrt{9} \\ \downarrow & & & & & \downarrow \\ =2 & & & & & =3 \end{array}$$

  Therefore, between 2 and 3.

- Somewhere between 2 and 3. I have no idea of the exact location, but it's closer to 2 than to 3.
- I used my calculator and found that $\sqrt{5} \approx 2.23$. Also $\sqrt{5} = 5^{1/2}$. To plot the point I found the midpoint between 2 and 3, then between 2 and 2.5, then plotted $\sqrt{5}$ roughly at 2.25.
- There are 5 whole numbers between 4 and 9 (perfect squares), and since 5 comes after 4 it will be 1/5 the way between 2 and 3.

In this response we note an example of 'overgeneralization of linearity' [10], a response that stems from what can be seen to hold true in linear relationships. In particular, the location of $\sqrt{5}$ is said to be obtainable using a linear interpolation between the two neighbouring perfect squares.

- Divide the section between 2 and 3 into 10 equal parts, find the two neighbouring tick-marks that correspond to just below and just above 5 when squared. Then divide this segment into 10 parts and repeat the process until you get better and better approximation.
- Closest perfect square is 4, $\sqrt{4} = 2$, so it is a little over 2. For greater accuracy, we would try more digits.

2.3×2.3	$= 5.29$	(too high)
2.2×2.2	$= 4.84$	(too low)
2.23×2.23	$= 4.9729$	(too low)
2.24×2.24	$= 5.0176$	(too high)
2.238×2.238	$= 5.008644$	(too high)
2.237×2.237	$= 5.004169$	(still too high)
2.236×2.236	$= 4.99696$	(too low)

4.3. *Function-graph approach*

This type of response was found among three participants. These approaches assume what is to be found; that is to say, they assume the availability of an accurate graph, from which the required length would be simply read off, instead of finding a way to construct such length. It should be noted that one of the three participants who offered this kind of response admitted his doubts about the validity of such an approach.

- Using functions, such as a sketch of $f(x) = x^2 - 5$ and then looking at the zero of this function $x^2 - 5 = 0$. A statement 'if my graph is **absolutely accurate**, I will find the exact location' accompanied this approach.

- Similar to the above, only using $f(x) = \sqrt{x}$ and then looking at the value of this function at $x = 5$ on the graph (the ordinate distance).

4.4. *Impossible?*

Some participants questioned the validity of the assignment. Most likely the word 'exact' triggered these kinds of responses.

- $\sqrt{1} = 1, \sqrt{2} \approx 1.4, \sqrt{3} \approx 1.7, \sqrt{4} = 2, \sqrt{5} \approx 2.3$
- I don't think you can find the exact location of $\sqrt{5}$ looking at the number line because it is a huge decimal form number. I do believe there is a way by using calculus, but I'm not sure how to do it.
- This is a trick question, as $\sqrt{5}$ is irrational, it **cannot be placed exactly** on the number line, because its digits are infinite.
- Can I find the exact location without knowing the rest of ∞ digits?
- You can't.
- Divide on calculator. There is no exact point like that.

4.5. *Real number line versus rational number line*

Since only 9 out of 46 prospective teachers (19.6%) were able to locate the $\sqrt{5}$ on the number line accurately, we investigated what may be the reason for these difficulties. A rather striking observation is that the vast majority of participants perceive the number line as a *rational* number line. It turns out that those arguing 'you can't' and those that used a more or less fine decimal approximation hold this perception. This can be concluded from the interviews where we probed for a precise, not approximate, location of $\sqrt{5}$. Under such demand, all participants that previously offered a decimal approximation later concluded that it cannot be done. In other words, the common opinion was that it must be rounded before it can be located.

Next, we look at a range of responses from the clinical interviews that may shed some light on why locating $\sqrt{5}$ is perceived to be so problematic.

(Responding to the question about whether $\sqrt{5}$ can be found on the number line precisely)

Anna: No, because we don't know the exact value, because 0.0 bigillion numbers ending with 5 is smaller than 0.0 bigillion numbers ending with 6. They're two different numbers, right, so **because it never ends we can never know the exact value**.

Kyra: Yeah, yeah, like you would never be able to finally say okay, this is where it is, because **there are still more numbers that you're reading off your irrational number**. But if you're using this scale of, you know, 1, between 1 and 2 is 2 cm or something, there's only so much precision that you can make with that point that you draw on there, **like I can't make it as precise as an irrational number** or, you know...

4.6. *Finding the precise location of rational numbers*

From these excerpts it is evident that part of the difficulty lies in the infinite digits. To confirm that it is not the irrationality itself, but the fact that there are infinitely many digits in the decimal expansion, the interviewer enquired about the precise placement of rational numbers.

Interviewer: How about 1/3, can you find the location of 1/3 on a number line?

Anna: On a number line?

Interviewer: Yeah . . .

Anna: Yeah, it would be, well okay you could divide, 1 divided by 3 and get that standard 0.3 repeating . . ., oh but that doesn't end either. Okay, (pause) um, I think because we know that the 3 will never change, do we really know, I don't know, because it repeats. Like how do we not know that in the one millionth decimal place it's a 4 or something, or 0 or another number, I don't know. But because that we assume that 3 repeats always, we can like sort of cut it off and round it.

Interviewer: Does that mean that we can't really find the exact location?

Anna: No, it's going to be somewhere in between 0.3 repeated and 0.3 repeated and then 4.

Interviewer: Somewhere in between?

Anna: But, no (laugh) I guess not, because it is a different number, like by stopping the repetition of a decimal you're like cutting off its value. Like you're assuming it has a specific value, when in actuality it doesn't have, in reality it doesn't.

Similary, the interview with William suggests that the number line is perceived in a limited sense, as containing only terminating decimals. That is to say, the number line is reduced to the common ruler as used in everyday life.

William: I can find approximate position probably, exact position like I'd probably have to round it off at some point, and then come up with an approximate position, 0.334, something like that, depending on how I could, you want it, let's say you want it accurate to the ten hundredth place, a 1,000th place, **I would round it off to that place** and . . .

Further in the interview there is a discussion about how this would be done, which leads into an inquiry about what it is that makes the breaking of the unit into 10 equal pieces easier than breaking it into three pieces.

William: That I know, I can put a ruler there and I know, that's easy. 10 can be done with a ruler, I can also do it with the compass . . . (here William tries showing that a unit can be broken into 10 equal parts using the compass, but does not succeed) . . . I don't know how, but I think there is a distinct possibility. The ruler is the simplest, and on ruler you don't see the, let's say 1 cm divided into 3 parts, that's again divided into 10 parts. Anytime I have to do that like 3.33, I would, I normally approximate, just approximate the 3 . . ..

It should be noted that William's understanding of irrational numbers was one of the weakest of all the prospective teachers that we interviewed. It would be very difficult to build the concept of irrationality from William's concept image of rational numbers. Although rational approximations are often sufficient for most practical applications, we see this as an extreme example of the number line being reduced to an ordinary ruler, where common fractions that have infinite repeating decimals cease to exist.

4.7. *From numerical to geometric approach*

As noted earlier the most common approach was using decimal approximation. The Pythagorean Theorem was seldom invoked by the question. We were curious to find out if this is just because it did not come to mind at the time the written part was administered, or whether there is a deeper issue. It turns out that although the prospective teachers are well acquainted with the theorem they would generally use it only for finding the unknown length in a given right triangle, and not for the purpose of constructing a desired length. In the excerpt that follows, the interviewer prompts Steve to consider a more geometric approach, and even shows how this can be done in the case of $\sqrt{2}$.

Interviewer: Okay, and next question. Um, how would you find the exact location of square root of 5 on the number line?

Steve: Okay, so again without using a calculator?

Interviewer: Yeah, without.

Steve: Um, what roughly find the, the two closest perfect squares so root 4 is 2, and root 9 is 3, so it's going to be somewhere between 2 and 3, so I guess I would then try like 2.2 and multiply it together to see whether it's 5, or whether it's lower, so I guess I'd just try different numbers, try multiplying different numbers together, and see how close to root 5...

Interviewer: That would be quite tedious without a calculator, right?

Steve: Yeah, yeah.

Interviewer: How about a more geometric approach?

[interviewer introduces the idea of finding $\sqrt{2}$ as a hypotenuse of an isosceles right-angle triangle with side of 1]

Steve: Oh okay, oh that's interesting.

Interviewer: Um hm, so I'm just trying to see if we can also do something geometric to find the exact location of square root of 5, because the other method would work perfectly fine, but it would be an approximation only and it would be quite tedious.

Steve: Um hm, um hm, so how can you come up with a square root of 5, um, (pause)

Interviewer: Always just say, you know, I'll want to skip that...

Steve: Well it's not that I want to skip, it would just take me a long time to think about number combinations that come to root 5...

Interviewer: In what way combinations are you talking about?

Steve: Well, you know, that works for the 45-45-90 triangle, root 2 does and you know, root 3 can work for the 60-30-90, but I'd have to, I guess I'd have to find out a **ratio that used root 5**. Yeah, I couldn't figure out the answer just looking at it... It would be really hard for me to do without a calculator.

Upon the prompting, Steve invokes the trigonometric ratios for some commonly used right triangles that students are expected to memorize in high school, failing to recognize that these trigonometric ratios have been derived using the Pythagorean Theorem in the first place. The fact is, only 20% of the participants were able to invoke their knowledge of the theorem in order to address the presented task. Furthermore, the eliciting questions at the time of the interviews still did not draw out or assist in evoking the theorem from the participants' concept image.

On this basis, we suggest that the knowledge of the Pythagorean Theorem is an inert kind of knowledge for a great majority of our prospective secondary mathematics teachers. We see this as a symptom of two general issues surrounding the present state of mathematics education: one, the trend of weakening of geometry in school curriculum, and two, the fragmentation of the curriculum. In other words, the knowledge required to be used in any particular unit of study is limited to what is explicitly approached in that unit and cross-topics connections are not encouraged. If some topic (we use the Pythagorean Theorem and its applications as an example) has already received its due share in the curriculum, it no longer needs to occupy students' time or minds. A limited exposure to geometry coupled with an infrequent need to apply the theorem may be responsible for the fact that the desired approach in responding to the construction question was found to be so rare.

4.8. *Precise location: What can be gained?*

Among those participants who were able to find the precise location of $\sqrt{5}$ we found there was a sense of security that such a number indeed existed. Their understanding seemed much more robust. Perhaps we could even say that the availability of a geometric representation aided them in the life cycle of concept development towards its final stage of encapsulation. This is in contrast with many others who offered the decimal approximation approach, where the number was seen as a process, stuck in its making forever. The following excerpt with Stephanie exemplifies this view.

Stephanie: Yeah. Okay, what I am thinking of, because somehow you can build this triangle and this triangle exists, this is another interpretation of the irrational number, so this segment represents the length of that hypotenuse, represents square root of 5, because this triangle exists. So it should be something what is, like we can touch, I don't know.

Finally, we present an excerpt from the interview with Claire, who communicated to us why she thinks teachers should not be satisfied with approximations.

Claire: Now, of course a point does not have dimensions. So on the number line you don't have actually the lead of the pencil, it's still a dimension, although it's not. So intuitively you can say yes, it's there, a number can be represented in this way . . . As an answer, if you have the construction with a compass, yes you assume that construction is exact and precise, yes, $1^2 + 1^2 = 2$ and square root of 2 is the exact representation of square root of 2 irrational number, not how we are used to say 1.41, which is an estimation, and approximate answer.

Interviewer: And what do you think is there, what's the importance of us um having a student understand this, you know, exact and approximate, when they always work with approximation? What is the value for you? Do you think they should learn about these things?

Claire: I still tend to believe that it's better to work with the exact value, rather than an estimation, instead of, I'm the person that I like to speak with the terminology in math, so saying that pi is 3.14 ends up, if you don't insist in elementary school in grade 7, 8 whatever, saying that it's not, it's only estimation of the number, but you explain the pi like being, you know, some, the lengths of the circle and whatever, I think it's very important the terminology here, to understand that they have a specific value.

Interviewer: Okay . . .

Claire: So I agree with not being careless about this. When it's exact value, it's exact value, when it's a rounding of a number, it's a rounding of a number in estimation.

At some point students need to become aware that there is a profound distinction between the exact value of an irrational number and its rational approximation. We suggest this is better done sooner than later. Our findings indicate deep misconception and apparent confusion of some students who do not understand the distinction between π and 22/7, as an example of an irrational number and its rational approximation. A similar confusion was reported previously by Arcavi *et al*. [7], where labelling 22/7 an irrational number was a common error. Also, students need to be aware of the effects of premature substitution of irrational values by their rational approximations in partial results during calculations, both in the sense that this complicates the calculations and creates problems of cumulative error. However, students' awareness will be hard to achieve if it is not within an active repertoire of their teachers.

5. Pedagogical considerations

A significant part of school curriculum is focused on the notion of number. The notion of a number line appears early in elementary school and aids in ordering numbers and introducing integers and operations with integers. As rational numbers are dense, the idea that they do not 'cover' the continuous number line presented a challenge to mathematicians. The formalization of this idea and formal definition of real numbers is presented through the introduction of Dedekind cuts and is beyond what is normally presented in school.

However, in the school curriculum today we expect students to accept, intuitively, the idea of one-to-one correspondence between real numbers and points on a number line, and rely on explaining real numbers as 'all the points on the number line'. It is important to be aware of the fact that 2500 years have passed from the 'discovery' of irrational numbers as lengths to the formal construction of the set of real numbers. It would be unreasonable to expect that what took centuries of mathematicians' work to develop could be acquired by students in a few sessions of classroom exposure.

The concept of an irrational number is inherently difficult; yet, understanding of irrational numbers is essential for the extension and reconstruction of the concept of number from the system of rational numbers to the system of real numbers. Therefore a careful didactical attention is essential for proper development of this concept.

We believe that emphasis on decimal representation of irrational numbers, be it explicit or implicit, does not contribute to the conceptual understanding of irrationality. And with irrational numbers one is faced with infinite decimal numbers of a special kind – numbers that cannot be written down or known fully. On this note, Stewart [11] challenges the wisdom of calling irrational numbers *real*;

that is, how can something be real if it cannot even be written down fully? In this sense, geometric representation should come almost as a relief in the process of learning about irrationals.

To be able to capture infinite decimals with something finite and concrete, and as simple as a point on the number line, even if this is only possible for a certain category of irrationals (constructible lengths), should help in taming the difficult notion of irrationality. Moreover, the geometric representation of irrational number may well turn out to be a very powerful and indispensable teaching tool for encapsulating a process into an object, especially in the case where the learner is on the verge of the reification stage in the development of the concept of irrationality. It is both accessible to the learner (required is the knowledge of the Pythagorean Theorem) and yet revealing of the idea that to every number there corresponds a (single) point on the number line. As such, it is our contention that placing more emphasis on the geometric representation of irrational numbers can aid students in two ways. First, they are likely to become more sensitive to the distinction between the irrational number and its rational approximation. Secondly, it is likely to help them encapsulate the concept of irrationality by drawing their attention to yet another representation of the object (point on the number line, an irrational distance from 0) and away from the never-ending process of construction in time, as often perceived through the infinite decimal representation. However, if teachers themselves do not possess the relevant content knowledge, achieving understanding in students is unlikely.

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